Gravitational Force Field in the Vicinity of the Earth-Moon Libration Points

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This paper is concerned with the determination of the shape and magnitude of the gravitational force field surrounding the libration points of the restricted Earth-moon system and indicates the propulsive velocity requirements for vehicle station-keeping in these regions. The effects resulting from solar perturbations and the eccentricty of the lunar orbit are indicated. It is found that around each libration point the curves generated by the intersections of the equipotential surfaces with the Earth-moon plane are oval, resembling concentric ellipses (of varying fineness ratio), and have their major axes aligned at right angles to the lines joining the respective libration points with the Earth. Expressed by the ratio of their major to minor axes, the shapes of these curves vary from 95:1 (a highly elongated oval) near the two equilateral-triangle points down to 3.4:1 near the translunar point and 2.7:1 near the cislunar point. The contours of constant acceleration and the direction of the vectors are shown in the figures. For convenience, the characteristic velocity requirements for one month's station-keeping are given. It can be seen that station-keeping is less costly around the equilateral-triangle point than near the linear libration points.

Nomenclature

A_u, A_v = $a_{ru}D^2/\mu_m$ and $a_{rv}D^2/\mu_m$, respectively
A_u', A_v' = $a_{ru}/\omega^2 D$ and $a_{rv}/\omega^2 D$, respectively
a_c = acceleration of libration point
a_I = acceleration of particle in inertial frame of reference
a_r = relative acceleration of particle with regard to rotating coordinates attached at libration point
a_{ru}, a_{rv} = components of a_r in the u and v directions
D = Earth-moon distance
F_G = total gravitational force
f_T = thrust acceleration
$\mathbf{i}_{Ec}$, $\mathbf{i}_{mc}$ = unit vector pointing from Earth or moon to libration point
$\mathbf{i}_{cg}$ = unit vector pointing from baricenter to the libration point
$\mathbf{i}_u$, $\mathbf{i}_v$ = unit vectors in u and v directions, respectively
k^2 = gravitational constant
k_1, k_2 = angles defined by Eq. (27)
l = distance from baricenter (Earth-moon center of mass) to libration point
R_1, R_2 = r_1/D, r_2/D
r, θ = polar coordinates of particle referred to libration point
r_1, r_2 = distances from Earth to particle and moon to particle, respectively
x_E = separation of Earth's center from Earth-moon baricenter
V = velocity requirement for station-keeping
μ = $\mu_m/(\mu_E + \mu_m) \simeq 1/82.45$
μ_E = Earth's gravitational constant = $k^2 M_E$
μ_m = moon's gravitational constant = $k^2 M_M$
ρ = r/D
ϕ, σ, δ = angles shown in Fig. 2
ψ = angle defined by Eq. (16)
ω = angular velocity of Earth-moon system $\cong$ 0.23 rad/day
ω_s = angular velocity of Earth around sun $\cong$ 0.0172 rad/day

Received by ARS August 2, 1962; revision received January 2, 1963. This research was sponsored by the U. S. Air Force under Project Rand. This is an abridgment of Rand Memo. RM-3150-PR. The views, conclusions, and recommendations expressed herein do not necessarily reflect the official views or policies of the U. S. Air Force.

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Introduction

THE problem of motion in the vicinity of the libration points of the Earth-moon system and the nature of the stability of this motion have received wide and careful attention in the classical as well as the more recent literature. Thus it is well known that the equilateral-triangle points are points of stable equilibrium, so that a particle initially placed near one of these points will continue to move in the immediate vicinity of these points, whereas the linear points lead to divergent (i.e., unstable) motion, causing the particle to recede further and further as time progresses. On the other hand, relatively little, if any, information has appeared in the literature concerning the magnitude, orientation, and shape of the force field in the vicinity of these points.

The nature and size of these so-called potential wells are of more than purely academic interest because they immediately determine the thrust requirements for a space vehicle that must maintain a constant position in Earth-moon space, but at a point displaced somewhat from the nominal libration point. It is the purpose of this paper to provide the necessary information about the force field surrounding the libration points. The material presented here is, except for the section dealing with the solar and eccentricity effects, essentially the same as that presented in Ref. 2.

General Relations

The inertial acceleration vector of a particle, expressed in terms of the acceleration experienced by the same particle in a moving and rotating frame of reference, u, v, w, is given by the relation

$$\mathbf{a}_I = \mathbf{a}_c + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathbf{v}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \mathbf{a}_r \quad (1)$$

where $\boldsymbol{\omega}$ is the angular velocity of the moving frame of reference.

This acceleration is caused by the action of the gravitational forces per unit particle mass

$$\mathbf{F}_G/m = -(\mu_E/r_1^2)\mathbf{i}_{r_1} - (\mu_m/r_2^2)\mathbf{i}_{r_2} \quad (2)$$

If this discussion is restricted to a coplanar model, one obtains for the relative acceleration of the particle, temporarily at rest in a uniformly rotating coordinate system

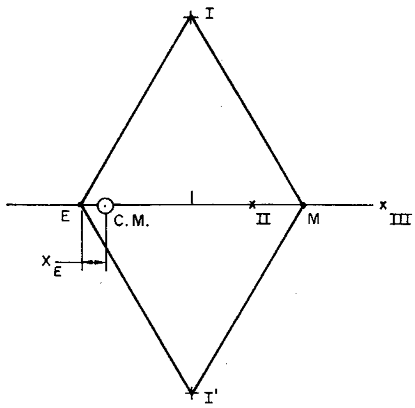

Fig. 1 Location of libration points

centered at point c, the expression

$$\mathbf{a}_r = a_{ru}\mathbf{i}_u + a_{rv}\mathbf{i}_v = -(\mu_E/r_1^2)\mathbf{i}_{r_1} - (\mu_m/r_2^2)\mathbf{i}_{r_2} - \mathbf{a}_c + \omega^2\mathbf{r} \quad (3)$$

The four libration points considered here are shown in Fig. 1.

The geometry for a particle located near one of the equilateral-triangle points is shown in Fig. 2. If the moving coordinate system is taken to be centered at this libration point, then

$$a_c = -r_c\omega^2\mathbf{i}_{cg} \quad (4)$$

The angular velocity ω of the coordinate system is defined as

$$\omega^2 = (\mu_E + \mu_m)/D^3 \quad (5)$$

where

$$\mu_E = k^2M_E \qquad \mu_m = k^2M_m$$

In order to maintain a fixed position in the u,v plane, a thrust acceleration $\mathbf{f}_r$ has to be applied by the vehicle's powerplant so as to cancel the relative gravitational acceleration of Eq. (3), i.e.,

$$\mathbf{f}_r + a_r = 0 \quad (6)$$

The particular form of Eq. (3) for each libration point is discussed in the following sections. These relations have been programmed on a digital computer, and contours of constant $|a_r|$ have been generated in the u,v plane.

For those cases where extreme accuracy is not required, it becomes possible to linearize Eq. (3) in the neighborhood of the libration points and by this means to generate very quickly the acceleration contours desired.

Equilateral-Triangle Libration Points

Exact Form

The leading point will be considered first. The position of the baricenter x_E as obtained from a moment balance is given by

$$x_E = [\mu_m/(\mu_m + \mu_E)]D = \mu D \quad (7)$$

From here

$$r_c^2 = D^2 + x_E^2 - Dx_E = D^2[1 + \mu^2 - \mu] \quad (8)$$

where

$$\mu \simeq 1/82.45$$

The distances r_1 and r_2 are given by

$$\begin{aligned} r_1^2 &= D^2 + r^2 - 2rD\cos(120° + \theta) \\ r_2^2 &= D^2 + r^2 - 2rD\cos(60° + \theta) \end{aligned} \quad (9)$$

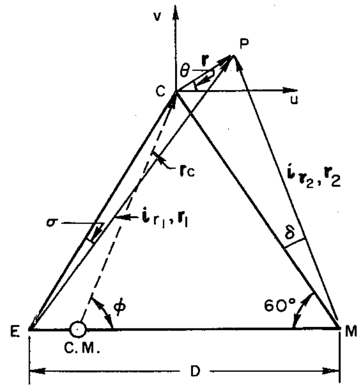

Fig. 2 Geometry of equilateral-triangle point

whereas the orientation of the unit vectors $\mathbf{i}_{r_1}$, $\mathbf{i}_{r_2}$, and $\mathbf{i}_{cg}$ is specified by

$$\begin{aligned} \mathbf{i}_{r_1} &= \cos(60° + \sigma)\mathbf{i}_u + \sin(60° + \sigma)\mathbf{i}_v \\ \mathbf{i}_{r_2} &= -\cos(60° + \delta)\mathbf{i}_u + \sin(60° + \delta)\mathbf{i}_v \\ \mathbf{i}_{cg} &= \cos\phi\mathbf{i}_u + \sin\phi\mathbf{i}_v \end{aligned} \quad (10)$$

The auxiliary angles σ, δ, and ϕ are defined as follows:

$$\sigma = -\sin^{-1}[(r/r_1)\sin(120° + \theta)](\text{rad}) \quad (11)$$

where

$$\sigma \geq 0 \qquad 60° \leq \theta \leq 240°$$
$$\sigma \leq 0 \qquad -120° \leq \theta \leq 60°$$

$$\delta = \sin^{-1}[(r/r_2)\sin(60° + \theta)](\text{rad}) \quad (12)$$

where

$$\delta \geq 0 \qquad -60° \leq \theta \leq 120°$$
$$\delta \leq 0 \qquad 120° \leq \theta \leq 300°$$

and

$$\phi = (\pi/3) + \sin^{-1}[(x_E/r_c)\sin 60°](\text{rad}) \quad (13)$$

If one now makes use of Eqs. (4–13) in Eq. (3) and nondimensionalizes the acceleration with respect to the magnitude of the lunar acceleration at the libration point μ_m/D^2, one arrives at the following expressions for the acceleration components:

$$\begin{aligned} A_u &= -\frac{1-\mu}{\mu R_1^2}\cos(60° + \sigma) + \frac{1}{R_2^2}\cos(60° + \delta) + \frac{R_c}{\mu}\cos\phi + \frac{\rho}{\mu}\cos\theta \\ A_v &= -\frac{1-\mu}{\mu R_1^2}\sin(60° + \sigma) - \frac{1}{R_2^2}\sin(60° + \delta) + \frac{R_c}{\mu}\sin\phi + \frac{\rho}{\mu}\sin\theta \end{aligned} \quad (14)$$

The nondimensional quantities employed in Eq. (14) are as follows:

$$A_u = (a_{ru}/\mu_m)D^2 \qquad R_1 = r_1/D \qquad R_c = r_c/D$$
$$A_v = (a_{rv}/\mu_m)D^2 \qquad R_2 = r_2/D \qquad \rho = r/D$$

The contours of constant acceleration are obtained by plotting ρ vs θ for selected values of

$$A = [A_u^2 + A_v^2]^{1/2} = \text{const} \quad (15)$$

whereas the direction of the acceleration vectors is found from the relation

$$\tan\psi = A_v/A_u \quad (16)$$

Some of the contours obtained from Eq. (15) are shown in Fig. 3. The foregoing developments for the leading equilateral-triangle point are immediately applicable also to the trailing point, provided that the sense of the $\mathbf{i}_v$ coordinate is reversed now and measures positive increase of the polar coordinate θ in the opposite (clockwise) direction.

Linearized Approximation

Retention of the linear terms of the Taylor series expansion around the position of libration point I leads to the following expressions for the relative acceleration vector:

$$\mathbf{a}_r = (\mu_E/D^3)[3(\mathbf{r}\cdot\mathbf{i}_{Ec})\mathbf{i}_{Ec} - \mathbf{r}] + (\mu_m/D^3)[3(\mathbf{r}\cdot\mathbf{i}_{mc})\mathbf{i}_{mc} - \mathbf{r}] + \omega^2\mathbf{r} \quad (17)$$

The unit vectors just used are shown in Fig. 2. Their direction cosines are indicated by

$$\begin{aligned}\mathbf{i}_{Ec} &= \cos60°\mathbf{i}_u + \sin60°\mathbf{i}_v \\ \mathbf{i}_{mc} &= -\cos60°\mathbf{i}_u + \sin60°\mathbf{i}_v\end{aligned} \quad (18)$$

When the expressions in brackets in Eq. (17) are evaluated and some manipulations performed, one obtains, after nondimensionalization with regard to $\omega^2 D$, the following two acceleration components:

$$\begin{aligned}A_u' &= 3\rho\cos60°[\cos(60° - \theta) - 2\mu\sin60°\sin\theta] \\ A_v' &= 3\rho\sin60°[\cos(60° - \theta) - 2\mu\cos60°\cos\theta]\end{aligned} \quad (19)$$

$$A_u' = a_{ru}/\omega^2 D \qquad A_v' = a_{rv}/\omega^2 D$$

The relation for ρ as a function of θ for any chosen value of $A' = (A_u'^2 + A_v'^2)^{1/2}$ is found to be

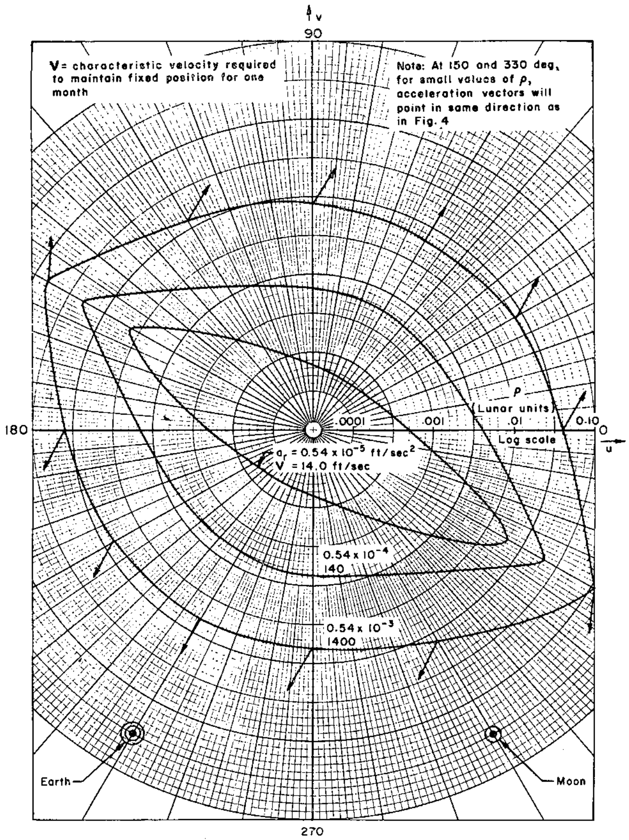

Fig. 3 Constant-acceleration profile, equilateral-triangle point I

$$\rho = \frac{A'}{3\{\cos^2(60° - \theta) - 2\mu\sin120°\cos(60° - \theta)\sin(60° + \theta) + \mu^2\sin^2 120°\}^{1/2}} \quad (20)$$

Equation (20) represents a doubly symmetric closed curve. The maximum value of ρ

$$\rho_{\max} \simeq A'/3\mu\sin120° \quad (21)$$

occurs at the one axis of symmetry, $\theta = -30°$, whereas the least value

$$\rho_{\max} \simeq A'/3 \quad (22)$$

occurs 90° away.

The ratio of major to minor axes of this elliptical curve is thus

$$\rho_{\max}/\rho_{\min} = 1/\mu\sin120° \simeq 82.45/0.866 = 95.21 \quad (23)$$

The orientation of the acceleration vectors is obtainable from the two components of Eq. (19) and is indicated schematically in Fig. 4.

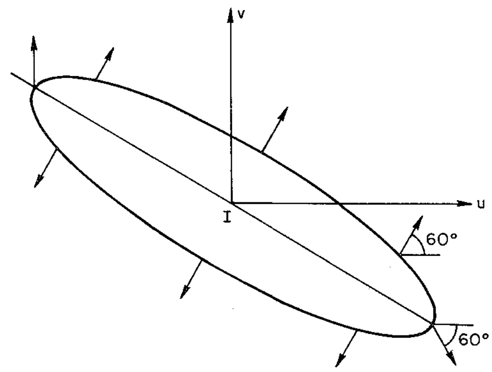

Fig. 4 Constant A' curve around equilateral-triangle point

Straight-Line Libration Points

Exact Form

The geometry for the straight-line libration points, with point II used for purposes of illustration, is shown in Fig. 5. The distances of points II and III from the c.g. of the system are[1]

$$l_{II} = 0.83702\,D \qquad l_{III} = 1.1556\,D \quad (24)$$

In a manner analogous to the one employed in Sec. III, one can obtain the two acceleration components in the form

$$\begin{aligned}A_u &= -[(1-\mu)/\mu R_1^2]\cos k_1 - (1/R_2^2)\cos k_2 + (l_{II}/\mu D) + (\rho/\mu)\cos\theta \\ A_v &= -[(1-\mu)/\mu R_1^2]\sin k_1 - (1/R_2^2)\sin k_2 + (\rho/\mu)\sin\theta\end{aligned} \quad (25)$$

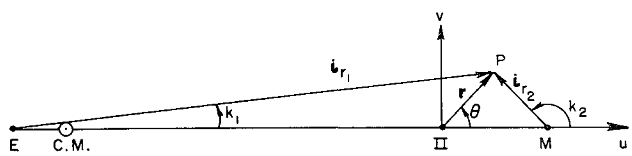

Fig. 5 Geometry for straight-line points

As before,

$$R_1 = r_1/D \qquad R_2 = r_2/D$$

For the present case, r_1 and r_2 are given by the relations

$$\begin{aligned}r_1^2 &= (l_{II} + x_E)^2 + r^2 + 2r(l_{II} + x_E)\cos\theta \\ r_2^2 &= (D - x_E - l_{II})^2 + r^2 - 2r(D - x_E - l_{II})\cos\theta\end{aligned} \quad (26)$$

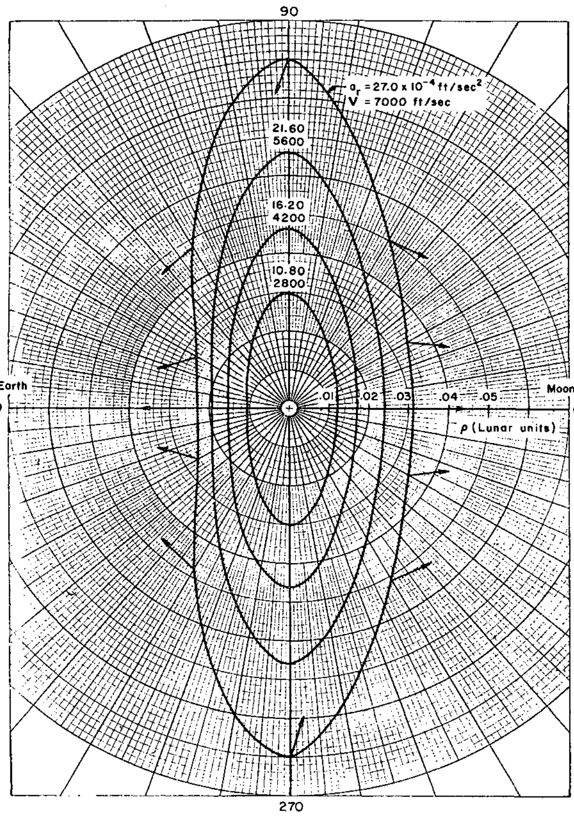

Fig. 6 Constant-acceleration profile, cislunar linear point II

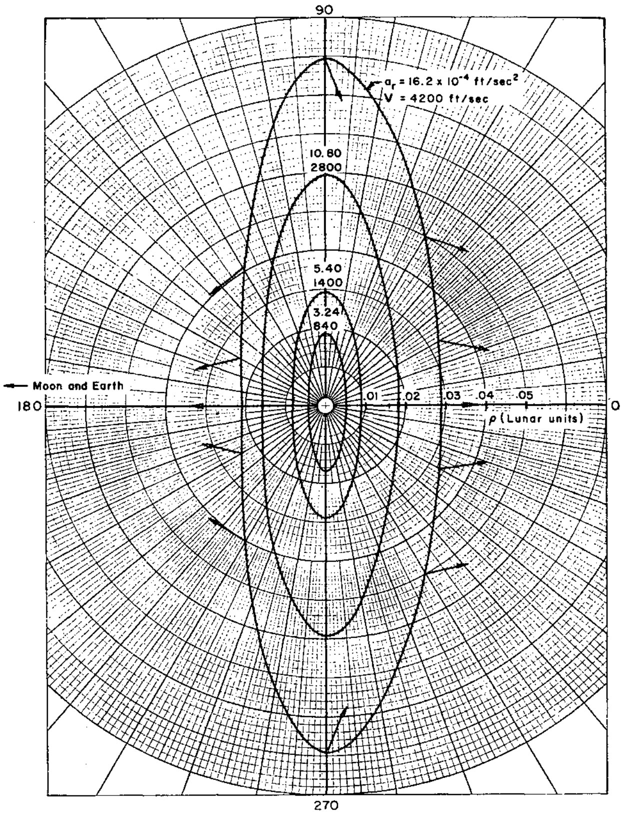

Fig. 7 Constant-acceleration profile, translunar linear point III

while

$$k_1 = \sin^{-1}[(r/r_1)\sin\theta]$$

and (27)

$$k_2 = \sin^{-1}[(r/r_2)\sin\theta]$$

The same relations apply also to point III if one simply replaces l_{II} by l_{III}. The results of the computations are presented in Figs. 6 and 7.

Linearized Approximation

The linearized treatment of Sec. III also is useful for the present case, except that now the region of applicability has to be restricted more severely because of the decreased magnitude of the distance from the moon to the libration points. Using the translunar point III as an example, one obtains

$$r_{mIII} = 1.1556D - [D - \mu D] = 0.1677D = \text{distance from moon to point III}$$

$$r_{EIII} = \mu D + 1.1556D = 1.1677D = \text{distance from Earth to point III}$$

$$a_{ru} = \left[\frac{2\mu_m r}{(0.1677D)^3} + \frac{2\,\mu_E r}{(1.1677D)^3} + \omega^2 r\right]\cos\theta$$

$$a_{rv} = \left[-\frac{\mu_m r}{(0.1677D)^3} - \frac{\mu_E r}{(1.1677D)^3} + \omega^2 r\right]\sin\theta \qquad (28)$$

Nondimensionalizing with respect to $\omega^2 D$ results in

$$A_u' = M\rho\cos\theta \qquad A_v' = N\rho\sin\theta \qquad (29)$$

where

$$M = \frac{2\mu}{(0.1677)^3} - \frac{2\mu}{(1.1677)^3} + \frac{2 + (1.1677)^3}{(1.1677)^3} \cong 7.385$$

$$N = \frac{\mu}{(1.1677)^3} - \frac{\mu}{(0.1677)^3} + \frac{(1.1677)^3 - 1}{(1.1677)^3} \cong -2.192$$

Equation (29) can be solved for ρ to give

$$\rho = A'/[M^2\cos^2\theta + N^2\sin^2\theta]^{1/2} \qquad (30)$$

The curve defined by Eq. (30) has two extremal values at $\theta = 0$ and $\theta = \pi/2$. In view of the difference in magnitude of M and N, it is obvious that the major axis of this elliptical curve is aligned at right angles to the Earth-moon line.

The ratio of major to minor axes is found to be

$$\rho_{\max}/\rho_{\min} = M/N \cong 3.36 \qquad (31)$$

The orientation of the acceleration vectors is indicated schematically in Fig. 8.

Solar Gravitational and Lunar Eccentricity Effects

The solar gravitational effect is of significance mainly at the equilateral-triangle points and can be estimated from the linear term of the Taylor expansion around the Earth-moon baricenter. In the fundamental plane, it gives rise to a secular gravitational acceleration of magnitude $\frac{1}{2}\ \omega_s{}^2D \cong 2.5 \times 10^{-5}$ ft/sec^2 and a periodic acceleration term of maximum amplitude $\frac{3}{2}\ \omega_s{}^2D = 7.5 \times 10^{-5}$ ft/sec^2 and angular velocity $\omega_0 = \omega - \omega_s = 0.2128$ rad/day which corresponds

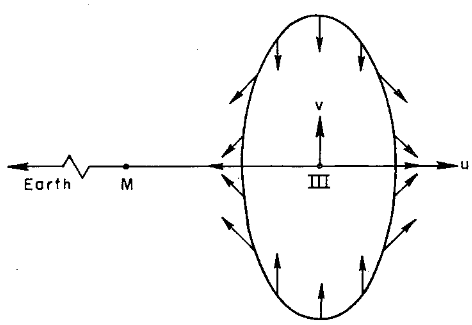

Fig. 8 Constant A' curve around linear point III

to a period of roughly 29.5 days. It can be shown[3] that the effect of the secular term can be canceled by a static displacement of the libration point (point I) through a distance of about 260 statute miles to the new coordinates ($u = -225.6$ miles, $v = -130$ miles), which then leaves only the periodic part to be counteracted by thrusting. The out-of-plane component is purely periodic, and its amplitude is 0.09 ($\approx \sin 5°$) times as large as the in-plane acceleration.

The effect of the eccentricity of the moon's orbit, $e \cong 0.055$, gives rise to an additional periodic perturbing acceleration that can be shown to lie between the limits of 3.06×10^{-5} ft/sec² and 1.53×10^{-5} ft/sec².

Concluding Remarks

The two-dimensional gravitational acceleration field in the neighborhood of the Earth-moon libration points has been investigated, and the shape of contours of constant relative acceleration have been determined. Of particular interest for purposes of powered station-keeping is the fact that, within a radius of 2500 miles around the libration points, thrust accelerations of not more than 10^{-3} ft/sec² will be adequate to maintain spaceships in a fixed position relative to the Earth-moon frame of reference.

Perturbing accelerations in the out-of-plane direction, experienced at points located above or below the Earth-moon plane, are of smaller magnitude than the in-plane components. Thus for the same 2500-mile displacement out of the fundamental plane, the acceleration should not excede roughly 10^{-4} ft/sec².

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[3] Milder, D. M., "Stabilizing a Trojan satellite" (unpublished work.)

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